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LETTER TO THE EDITOR

On the statistical averaging procedure for the refractive index of matter waves

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Abstract
A discrepancy concerning the statistical averaging procedure is resolved for the refractive index of matter waves travelling in a dilute gas.

In determining the index of refraction for matter wave propagation in a dilute gas, it is necessary to perform a statistical average with respect to all microscopic parameters of the medium. Taking account of the thermal motion of the medium has led to a discrepancy regarding the velocity distribution function of the gas particles [1–3]. The discrepancy is attributed to a dragging or Fizeau effect [2] caused by the motion of the particles in the medium. At low values of the relative velocity, the difference between the distribution functions given in [1–3] with the correct one given in [4] becomes significant. Here, we clarify the issues that lead to the distribution function given in [4].

The application of multiple scattering theory to the present problem has been described previously [4]. It was claimed [2, 3] that the distribution function used in [4] to compute the refractive index is incorrect due to neglect of the vectorial nature of the wavevector. We show that these claims are unfounded. The dispersion equation for a matter or electromagnetic wave propagating in a medium of independent scatterers is given by [5, 6]

$$k_L'^2 = k_L^2 + 4\pi N\langle f_L(k_L, 0)\rangle \tag{1}$$

where N is the number density of the medium, $\vec{k}_L$ and $\vec{k}_L'$ are the respective wavevectors in the vacuum and medium, and $\langle f_L(k_L, 0)\rangle$ is the average forward-scattering amplitude in the laboratory frame. The square of the refractive index is defined by

$$[n(k_L)]^2 = \frac{k_L'^2}{k_L^2} = 1 + \frac{4\pi N}{k_L^2}\langle f_L(k_L, 0)\rangle. \tag{2}$$

From the dispersion equations (1) and (2), we see that the index of refraction does not depend on the direction of the wavevector. This makes perfect sense when we consider that a spherical wave having no preferred direction has the same value of refractive index as a plane wave. For a dilute gas, the index of refraction is given by [4]

$$n(k_L) = 1 + \frac{2\pi N}{k_L}\left\langle \frac{f(k, 0)}{k}\right\rangle. \tag{3}$$

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The forward-scattering amplitude $f(k, 0)$ is calculated in the centre-of-mass (CM) frame for two particles with relative momentum k. A key step in the derivation of equation (3) is the use of the Lorentz invariant ratio [7, 8]

$$\frac{f_L(k_L, 0)}{k_L} = \frac{f(k, 0)}{k}. \tag{4}$$

The invariance of the imaginary part of equation (4) is clear from the optical theorem. The invariance of the real part is easily demonstrated using the Born approximation

$$\frac{f_L(k_L, 0)}{k_L} = \frac{f(k, 0)}{k} = -\frac{1}{2\pi v} \int V(\vec{r})\,\mathrm{d}\vec{r} \tag{5}$$

where the relative velocity v and the volume integral of the potential V are invariant with respect to Galilean transformations.

The average in equation (3) is taken with respect to the distribution function

$$\rho(k_L, k) = \frac{m_T + m_P}{k_L} \sqrt{\frac{2\beta}{\pi m_T}} \sinh\left(\frac{\beta m_T k_L k}{\mu m_P}\right) \exp\left\{-\beta\left[\frac{k^2}{2\mu}\left(1 + \frac{m_T}{m_P}\right) + \frac{k_L^2}{2m_P}\left(\frac{m_T}{m_P}\right)\right]\right\}. \tag{6}$$

Here μ is the reduced mass $m_T m_P/(m_T + m_P)$ with m_P and m_T the respective masses of the projectile and target atoms, and β is the inverse of the Boltzmann constant times the temperature of the target gas. This distribution function takes into account different values and directions of the relative velocities. This may be seen by examining the derivation of the formula, which begins by integrating the energy conserving delta function over a Maxwellian velocity distribution

$$\rho(k_L, k) = \int \delta\left[\frac{\mu(\vec{v}_P - \vec{v}_T)^2}{2} - \frac{k^2}{2\mu}\right] f_M(\vec{v}_T)\,\mathrm{d}\vec{v}_T \tag{7}$$

$$f_M(\vec{v}_T) = \left(\frac{\beta}{2\pi}\right)^{\frac{3}{2}} \exp\left(-\frac{1}{2}\beta m_T v_T^2\right), \tag{8}$$

where $\vec{v}_P$ is the velocity of the projectile atom and $\vec{v}_T$ is the velocity of the target gas atom. The integration is easily performed in spherical coordinates using the change of variable

$$u = \frac{\mu}{2}(v_P^2 + v_T^2) - \mu v_P v_T \cos\theta - \frac{k^2}{2\mu} \tag{9}$$

$$\mathrm{d}u = \mu v_P v_T \sin\theta\,\mathrm{d}\theta \tag{10}$$

which yields

$$\rho(k_L, k) = \frac{2\pi}{\mu v_P} \iint \delta(u)\,\mathrm{d}u\, f_M(\vec{v}_T) v_T\,\mathrm{d}v_T = \frac{2\pi}{\mu v_P} \int_{|v_P - k/\mu|}^{|v_P + k/\mu|} f_M(\vec{v}_T) v_T\,\mathrm{d}v_T. \tag{11}$$

Integration of equation (11) produces the distribution function given by equation (6). It is clear from this derivation that the vectorial nature of the projectile and target velocities has been properly taken into account. The required average in equation (3) is given by

$$\left\langle \frac{f(k, 0)}{k} \right\rangle = \frac{1}{\mu} \int_0^\infty f(k, 0)\rho(k_L, k)\,\mathrm{d}k \tag{12}$$

where

$$1 = \int_0^\infty \rho(k_L, k)\,\mathrm{d}\left(\frac{k^2}{2\mu}\right) = \frac{1}{\mu} \int_0^\infty k\rho(k_L, k)\,\mathrm{d}k \tag{13}$$

normalizes the distribution function. The origin of the discrepancy [2, 3] may be traced to the incorrect transformation equation

$$\vec{k}_L(n_L - 1) = \vec{k}(n_{CM} - 1) \tag{14}$$

where n_{CM} was defined to be the index of refraction in the CM frame [2]. For non-parallel $\vec{k}_L$ and $\vec{k}$, it was argued that a dragging or Fizeau effect must be taken into account [2]. We do not believe that this is the correct interpretation. In particular, we disagree with the use of equation (14) and believe that it is equation (4) that should be used when transforming between reference frames. Because equation (3) is the leading order term of a Taylor series expansion for the square root of n^2, it is incorrect to view equation (14) as a generalized vector version of equation (3). It is not even clear whether it is meaningful to define an index of refraction in the CM frame because each pair of colliding particles has its own CM. However, it is clear that equation (4) is invariant under transformation and forms a natural foundation for the averaging procedure. Furthermore, the distribution function given by equation (6), when multiplied by the wavenumber for relative motion, reduces to a Maxwellian distribution in the limit of zero beam velocity

$$k \lim_{k_L \to 0} \rho(k_L, k) \sim k^2 \exp\left(-\beta \frac{m_T k^2}{2\mu^2}\right). \tag{15}$$

The distribution function given by Vigue and co-workers [2] and also the one given by Leo *et al* [3] do not reproduce this physically correct limiting behaviour. If the target gas atoms in the medium are cooled to the limit of zero temperature, then the distribution function (6) becomes a delta function

$$\lim_{T \to 0} \rho(k_L, k) = \frac{\mu}{k} \delta\left(k - \frac{\mu}{m_P} k_L\right) \tag{16}$$

and

$$f_L(k_L, 0) = \frac{m_P}{\mu} f\left(\frac{\mu}{m_P} k_L, 0\right). \tag{17}$$

Equation (17) was derived for two-body amplitudes in the classical $T \to 0$ limit of the medium. The medium is assumed to be sufficiently dilute that a statistical coarse-graining procedure remains adequate when the de Broglie wavelength of the target atom is large. The result also applies in the $T \to 0$ limit for a mean field description of the medium [9].

Atom interferometry experiments [10] perform simultaneous measurements of the phase shift and the attenuation of the interfering amplitude. Therefore, they are most sensitive to the ratio

$$R(k_L) = \frac{\mathrm{Re}[n(k_L) - 1]}{\mathrm{Im}[n(k_L)]} = \frac{\int_0^\infty \mathrm{Re}[f(k, 0)]\rho(k_L, k)\,\mathrm{d}k}{\int_0^\infty \mathrm{Im}[f(k, 0)]\rho(k_L, k)\,\mathrm{d}k}. \tag{18}$$

It is convenient to define

$$R_0 = \lim_{T \to 0} R(k_L) = \frac{\mathrm{Re}[f(\frac{\mu}{m_P} k_L, 0)]}{\mathrm{Im}[f(\frac{\mu}{m_P} k_L, 0)]}. \tag{19}$$

The $k \to 0$ limiting behaviour of $R_0(k)$ may be used to determine the parameters in the effective range expansion [11]

$$R_0(k) \sim -(a_s k)^{-1} + \tfrac{1}{2} r_e k \tag{20}$$

where a_s is the scattering length and r_e is the effective range of the potential. The optical theorem may be used together with (20) to show that the scattering length is given by

$$a_s = -\lim_{k \to 0} \mathrm{Re}[f(k, 0)]. \tag{21}$$

If we divide the scattering length by the relative velocity v, we obtain a characteristic time of collision

$$t = \frac{a_s}{v} = -\mu \lim_{k\to 0}\left[\frac{\mathrm{Re}\, f(k,0)}{k}\right] = -\mu \lim_{k_L\to 0}\left[\frac{\mathrm{Re}\, f_L(k_L,0)}{k_L}\right]. \tag{22}$$

This time is invariant with respect to reference frame and is related to the lifetime Q of the scattering state [7]

$$Q = 2\frac{\mathrm{d}\delta}{\mathrm{d}\epsilon} = -2t = -\frac{2a_s}{v}. \tag{23}$$

The formulation given in [2] does not reproduce this result. Generally, the differences between the distribution functions given in [1–3] and the correct one given in [4] are magnified for low temperatures and beam velocities. Therefore, in these limiting cases, the error introduced through an incorrect choice of distribution function would be severe.

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