1 Gamma Function

\[ \Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt \] (1)

Exercise: use integration by parts to show that \( \Gamma(x + 1) = x \Gamma(x) \)

The so-called “Pochammer symbol” is defined by \((x)_n = \frac{\Gamma(x + n)}{\Gamma(x)}\).

Exercise: derive the recursion relation \((x)_n = (x + n - 1)(x)_{n-1}\) and compute a few iterations beginning with \((x)_0 = 1\).

Exercise: assuming that a code exists for computing \(\Gamma(x)\), write a code to compute \(1/\Gamma(x)\) that is valid for all \(x\) (including \(x = 0\)).

2 Beta Function

\[ B(x, y) = \int_0^1 t^{x-1} (1 - t)^{y-1} dt \] (2)

Exercise: substitute \(t = v/(1 + v)\) and show that

\[ B(x, y) = \int_0^\infty v^{x-1} (1 + v)^{-x-y} dv \] (3)

Exercise: using equation (1) show that

\[ \Gamma(x + y) = (1 + v)^{x+y} \int_0^\infty e^{-t(v+1)} t^{x+y-1} dt \] (4)

Exercise: multiply both sides of equation (4) by \(v^{x-1}(1+v)^{-x-y}\) and integrate between 0 and \(\infty\) to show

\[ \Gamma(x + y) \int_0^\infty v^{x-1} (1 + v)^{-x-y} dv = \int_0^\infty v^{x-1} dv \int_0^\infty e^{-t(v+1)} t^{x+y-1} dt \] (5)

Exercise: reverse the order of integration on the right hand side of equation (5) and substitute \(v = u/t\) and \(dv = du/t\) to show

\[ B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)} \] (6)
3 Hypergeometric Function

\[ 2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-tz)^{-a} \, dt \quad (7) \]

Exercise: using equations (2), (6), and (7) show that

\[ 2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad (8) \]

Exercise: use the identities

\[ (1+z)^m = \sum_{n=0}^{\infty} \binom{m}{n} z^n \quad |z| < 1 \quad (9) \]

and

\[ \left( \frac{-m}{n} \right) = (-1)^n \binom{m+n-1}{n} \quad (10) \]

to show that

\[ (1-tz)^{-a} = \sum_{n=0}^{\infty} \binom{a+n-1}{n} t^n z^n \quad (11) \]

Exercise: substitute equation (11) into equation (7) and show that

\[ 2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n \quad (12) \]

Equations (7) and (12) are called the integral and series representations of the hypergeometric function.

Exercise: use the ratio test to show that the series representation given by equation (12) converges for \(|z| < 1\).

Exercise: use the series representation to show

\[ \frac{d}{dz} 2F_1(a, b; c; z) = \left( \frac{ab}{c} \right) 2F_1(a+1, b+1; c+1; z) \quad (13) \]
4 Differential Equation

\[ z(1 - z) \frac{d^2u}{dz^2} + [c - (a + b + 1)z] \frac{du}{dz} - abu = 0 \] \hspace{1cm} (14)

Exercise: show that the series representation given by equation (12) is a solution of the differential equation (14).

Exercise: a second-order linear differential equation must also have a second independent solution. Show that \( z^{1-c} \binom{2}{a} F_1(a - c + 1, b - c + 1; 2 - c; z) \) is a solution of the differential equation (14).

Exercise: substitute \( z = 1 - z', \ a = a', \ b = b', \) and \( c = 1 + a' + b' - c' \) into equation (14) and show that the differential equation does not change.

Exercise: substitute \( z = 1/z', \ a = a', \ b = 1 + a' - c', \ c = 1 + a' - b', \) and \( u = (-z')^{-a'} u' \) into equation (14) and show that it does not change.

Exercise: substitute \( z = 1/z', \ a = 1 + b' - c', \ b = b', \ c = 1 + b' - a', \) and \( u = (-z')^{-b'} u' \) into equation (14) and show that it does not change.

Exercise: substitute \( z = 1 - z', a = b' - c' + 1, b = a' - c' + 1, c = a' + b' - c' + 1, \) and \( u = (1 - z)^{c-a-b} u' \) into equation (14) and show that it does not change.

The above exercises demonstrate that we have a total of six independent solutions. We will define these as follows:

\[ u_1 = \binom{2}{a} F_1(a, b; c; z) \] \hspace{1cm} (15)

\[ u_2 = \binom{2}{a} F_1(a, b; a + b + 1 - c; 1 - z) \] \hspace{1cm} (16)

\[ u_3 = (-z)^{-a} \binom{2}{a} F_1(a, a - c + 1; a - b + 1; 1/z) \] \hspace{1cm} (17)

\[ u_4 = (-z)^{-b} \binom{2}{a} F_1(b, b - c + 1; b - a + 1; 1/z) \] \hspace{1cm} (18)

\[ u_5 = z^{1-c} \binom{2}{a} F_1(a - c + 1, b - c + 1; 2 - c; z) \] \hspace{1cm} (19)

\[ u_6 = (1 - z)^{c-a-b} \binom{2}{a} F_1(c - a, c - b; c - a - b + 1; 1 - z) \] \hspace{1cm} (20)
5 Transformation Theory

Exercise: substitute \( s = 1 - t \) into equation (7) and show

\[
_{2}F_{1}(a, b; c; z) = (1 - z)^{-a} _{2}F_{1}(a, c - b; c; \frac{z}{1-z})
\]  

(21)

Exercise: use equation (21) to show

\[
u_2 = z^{-a} _{2}F_{1}(a, a - c + 1; a + b - c + 1; 1 - 1/z)
\]  

(22)

\[
u_3 = (1 - z)^{-a} _{2}F_{1}(a, c - b; a - b + 1; (1 - z)^{-1})
\]  

(23)

\[
u_4 = (1 - z)^{-b} _{2}F_{1}(b, c - a; b - a + 1; (1 - z)^{-1})
\]  

(24)

\[
u_6 = z^{a-c}(1 - z)^{c-a-b} _{2}F_{1}(c - a, 1 - a; c - a - b + 1; 1 - 1/z)
\]  

(25)

Any one of the six solutions may be expressed as a linear combination of two other solutions, e.g.

\[
u_1 = A \nu_2 + B \nu_6
\]  

(26)

\[
u_1 = C \nu_3 + D \nu_4
\]  

(27)

Exercise: set \( z = 1 \) in equations (26) and (27) to show that the coefficients are given by

\[
A = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}
\]  

(28)

\[
B = \frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)}
\]  

(29)

\[
C = \frac{\Gamma(c)\Gamma(b - a)}{\Gamma(b)\Gamma(c - a)}
\]  

(30)

\[
D = \frac{\Gamma(c)\Gamma(a - b)}{\Gamma(a)\Gamma(c - b)}
\]  

(31)
6 Generalized Hypergeometric Series

\[ mF_n(x_1, x_2, \ldots, x_m; y_1, y_2, \ldots, y_n; z) = \sum_{k=0}^{\infty} \frac{(x_1)_k(x_2)_k \cdots (x_m)_k}{(y_1)_k(y_2)_k \cdots (y_n)_k} \left( \frac{z^k}{k!} \right) \]  

\[ (32) \]

7 Special Cases

\[ _2F_1(1, 1; 2; -z) = z^{-1} \ln(1 + z) \]  

\[ (33) \]

\[ _2F_1(-a, b; b; -z) = (1 + z)^a \]  

\[ (34) \]

\[ _2F_1(0.5, 1; 1.5; -z^2) = z^{-1} \tan^{-1}(z) \]  

\[ (35) \]

\[ _2F_1(2a, a + 1; a; z) = (1 + z)/(1 - z)^{2n+1} \]  

\[ (36) \]

\[ _1F_1(a, a; z) = \exp(x) \]  

\[ (37) \]

\[ _1F_1(0.5, 1.5; -z^2) = z^{-1} \operatorname{erf}(z) \]  

\[ (38) \]

\[ _1F_1(-n, 1; z) = \operatorname{Ln}(z) \]  

\[ (39) \]

\[ _2F_1(-n, n + 1; 1; \frac{1-z}{2}) = \operatorname{Pn}(z) \]  

\[ (40) \]

\[ _2F_1(-n, n; \frac{1}{2}; \frac{1-z}{2}) = \operatorname{Tn}(z) \]  

\[ (41) \]

\[ _1F_1(n + \frac{1}{2}; 2n + 1; 2z) = n! \exp(z) \left( \frac{z}{2} \right)^{-n} \operatorname{In}(z) \]  

\[ (42) \]

\[ _1F_1(n + \frac{1}{2}; 2n + 1; 2iz) = n! \exp(iz) \left( \frac{z}{2} \right)^{-n} \operatorname{Jn}(z) \]  

\[ (43) \]

\[ _1F_1(-n; \frac{1}{2}; z^2) = (-1)^n \left[ \frac{1}{(2n)!} \right] \operatorname{H}_{2n}(z) \]  

\[ (44) \]